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Gauge-invariant canonical quantisation of the electromagnetic field and duality transformations

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Abstract. The free electromagnetic field is canonically quantised in a gauge-invariant way by interpreting the Fourier coefficients of the magnetic induction field $\boldsymbol{B}$ as generalised coordinates, and the coefficients of the electric field $\boldsymbol{E}$ as their conjugate momenta. The usual commutation relations among the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ are obtained. A canonical transformation, corresponding to a rotation in generalised phase space, is made on the Fourier coefficients. This transformation is shown to give a duality transformation on the electric and magnetic fields. The free-field Maxwell equations and the commutation relations are invariant under duality transformations. However, if interactions are introduced, the invariance under duality transformations is broken, and the original canonical theory should be used.

1. Introduction

In the usual approach (Heitler 1954) to the quantisation of the free electromagnetic field the gauge of the electromagnetic potentials is first fixed (Fermi 1931) in either the radiation (Coulomb) gauge or the Lorentz gauge (Heisenberg and Pauli 1929, Gupta 1950, Bleuler 1950). If the radiation gauge is used, then a Fourier expansion of the transverse vector potential is made. When the Hamiltonian is expressed in terms of the vector potential, it reduces to a sum of uncoupled harmonic oscillator Hamiltonians. The harmonic oscillators are then canonically quantised (Sakurai 1967, Bjorken and Drell 1965). If the Lorentz gauge (Wentzel 1949) is used for quantisation, subsidiary conditions must be imposed and an indefinite metric used to avoid contradictions (Mandl 1959, Schweber 1961). It must then be shown that the two procedures yield the same results, so that gauge invariance is ensured (Haller 1973, 1975, Sohn and Haller 1977, Zumino 1960, Strocchi and Wightman 1974).

In order to avoid the problems associated with using a particular gauge for quantisation, a number of authors have developed gauge-independent or gauge-invariant quantisation procedures (Dirac 1955, Bergmann 1956, Goldberg and Marx 1968, Marx 1970, 1972, Rzążewski and Wodkiewicz 1980, Woolley 1980). Non-local (Belinfante and Lomont 1951, Belinfante 1951a, b, Goldberg 1958, 1965) or path-dependent (Mandelstam 1962) methods, as well as non-canonical methods (Sharp 1968, Menikoff and Sharp 1977), have been suggested. The idea of using the electric and magnetic fields themselves instead of the potentials in the quantisation of the free electromagnetic field goes back to Jordan and Pauli (1928). Kramers' treatment (1958) uses a complex vector in which the real part is the electric field and the imaginary part is the magnetic field. His approach is similar to one previously used by Pauli (1933).

Power (1964) developed a manifestly gauge-invariant canonical quantisation procedure which directly uses the electric and magnetic fields. The Hamiltonian density for the free electromagnetic field is $\frac{1}{2}(E^2+B^2)$, where $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic induction field. By expanding $\boldsymbol{B}$ and $\boldsymbol{E}$ in normal modes, and interpreting the expansion coefficients of $\boldsymbol{B}$ as generalised coordinates and the expansion coefficients of $\boldsymbol{E}$ as the conjugate momenta, he was able to show that the Hamiltonian for the free electromagnetic field reduces to the Hamiltonian for a countable set of uncoupled harmonic oscillators. The oscillators were quantised in the canonical way. Scully and Lamb (1967), on the other hand, turn the procedure around. They associate the expansion coefficients of $\boldsymbol{E}$ with the generalised coordinates and the expansion coefficients of $\boldsymbol{B}$ with the conjugate momenta (Sargent *et al* 1974).

Both Power and Scully and Lamb use the Hamiltonian of the free electromagnetic field as their starting points. They both assume a relationship between the canonical momentum and generalised velocity. However, in order to define the conjugate momentum, it is necessary to use the Lagrangian. Then the Hamiltonian is defined by a Legendre transformation. Power introduces the standard Lagrangian density (Jackson 1975, Kobe 1981) $\frac{1}{2}(E^2-B^2)$ only *after* he obtains the equations of motion. Scully and Lamb never discuss the Lagrangian density. The approaches used by Power and Scully and Lamb are thus not strictly canonical.

If the usual Hamiltonian is to be derived from the canonical procedure, the choice of generalised coordinates made by Scully and Lamb requires the use of the non-standard Lagrangian density (Jackson 1975) $\frac{1}{2}(B^2-E^2)$. Although this Lagrangian density gives Maxwell's equations for the free field, its use in the total Lagrangian of a system of fields and charged matter does not give Maxwell's equations. Scully and Lamb introduce interactions in the electric dipole approximation by using $-q\boldsymbol{E}\cdot\boldsymbol{r}$ for the interaction Hamiltonian for a particle of charge q and displacement $\boldsymbol{r}$. A similar procedure was followed by Savolainen and Stenholm (1972) and Stenholm (1973), who also use the electric dipole approximation. They use the electric displacement vector $\boldsymbol{D}$ and the magnetic field strength $\boldsymbol{H}$, however.

In this paper the ambiguity as to whether the generalised coordinates should be associated with the Fourier coefficients of $\boldsymbol{E}$ or $\boldsymbol{B}$ is resolved by examining the Lagrangian formulation. The standard Lagrangian density for the electromagnetic field is $\frac{1}{2}(E^2-B^2)$ (Jackson 1975, Kobe 1981). When this expression is compared with the Lagrangian of classical mechanics, which is kinetic energy minus potential energy, the $\frac{1}{2}E^2$ plays the role of a kinetic energy and the $\frac{1}{2}B^2$ plays the role of a potential energy. Thus it seems natural from the Lagrangian point of view to associate the expansion coefficients of $\boldsymbol{B}$ with the generalised coordinates, and the expansion coefficients of $\boldsymbol{E}$ with the generalised velocities. When a canonical momentum is defined, the expansion coefficients of $\boldsymbol{E}$ are then related to momentum. This assignment is also in keeping with the classical theory of the electromagnetic field, based on potentials, where the momentum conjugate to the generalised coordinate $\boldsymbol{A}$ is $-\boldsymbol{E}$ (Kobe 1981). Without using potentials, we develop the theory for the free electromagnetic field in this paper using a strictly canonical formulation starting from the standard Lagrangian, defining canonical momentum, deriving the Hamiltonian, and obtaining the equations of motion. The kinematical equations give Faraday's law, while the dynamical equations give the Ampère–Maxwell law.

The theory of Power for the free electromagnetic field is related to the theory of Scully and Lamb by a canonical transformation on the generalised coordinates and momenta, which is a rotation in generalised phase space. The canonical transformation is shown to be equivalent to a duality transformation (Jackson 1975, p 252) on the electric and magnetic fields. Under a duality transformation the free-field Maxwell equations are invariant, as are the commutation relations between the field components. For the *free* electromagnetic field, the procedure of Scully and Lamb is equivalent to that of Power.

When interactions are present the two approaches are no longer equivalent. In the approach of Power (1964), Faraday's law is unchanged since it is kinematical. The Ampère–Maxwell law with the current density is obtained as the dynamical law. On the other hand, in the approach of Scully and Lamb (1967) and Sargent *et al* (1974) the free-field Ampère–Maxwell law is the kinematical equation while Faraday's law is the dynamical equation. When interactions are present, Faraday's law is no longer satisfied. Therefore we conclude that when the electromagnetic field is interacting with charged matter, the correct approach is the one of Power based on the standard Lagrangian density for the electromagnetic field.

In § 2 the Lagrangian formalism for the free electromagnetic field is developed for the Fourier coefficients of $\boldsymbol{B}$ as generalised coordinates and of $\boldsymbol{E}$ as generalised velocities. The corresponding canonical formalism is given in § 3 which includes both the Hamiltonian formalism and Poisson brackets. Canonical quantisation is performed in § 4. In § 5 canonical transformations corresponding to rotations in generalised phase space are made on the Fourier coefficients. These transformations are shown to give a duality transformation on the electric and magnetic fields in § 6. Finally the conclusion is given in § 7.

2. Lagrangian formulation

A Lagrangian formulation of the free electromagnetic field is given in this section. It differs from the usual Lagrangian formulations (Heitler 1954) in that potentials are not introduced. In this approach Gauss's law, the condition that magnetic monopoles do not exist, and Faraday's law are kinematical equations, while the Ampère–Maxwell law is obtained as a dynamical equation.

The standard gauge-invariant Lagrangian for the free electromagnetic field is (Jackson 1975, Kobe 1981, Sakurai 1967, pp 12–5)

$$L=\tfrac{1}{2}\int \mathrm{d}^3x\,(E^2-B^2), \tag{2.1}$$

in Lorentz–Heaviside units ($c=1$). The form of this Lagrangian suggests a natural way to introduce the generalised coordinates and velocities. When equation (2.1) is compared with the Lagrangian of classical mechanics, the location of the minus sign in equation (2.1) suggests that $\frac{1}{2}E^2$ be interpreted as kinetic energy and $\frac{1}{2}B^2$ be interpreted as potential energy. Hence it is natural to associate $\boldsymbol{B}$ with generalised coordinates and $\boldsymbol{E}$ with generalised velocities.

A Fourier expansion of the magnetic induction field $\boldsymbol{B}$ in a cubic volume V can be made,

$$\boldsymbol{B}(\boldsymbol{x},t)=V^{-1/2}\sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2} k\hat{\boldsymbol{\varepsilon}}^{(\lambda)}(\boldsymbol{k})q_{\boldsymbol{k}\lambda}(t)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}), \tag{2.2}$$

in which the coefficients $q_{\boldsymbol{k}\lambda}$ are interpreted as generalised coordinates. The unit polarisation vectors $\hat{\boldsymbol{\varepsilon}}^{(\lambda)}(\boldsymbol{k})$ for $\lambda=1,2$ are chosen such that $\hat{\boldsymbol{\varepsilon}}^{(1)}(\boldsymbol{k})$, $\hat{\boldsymbol{\varepsilon}}^{(2)}(\boldsymbol{k})$ and

$\hat{k} = \boldsymbol{k}/k$ form a right-handed orthonormal basis of three vectors which satisfy

$$\hat{\varepsilon}^{(\lambda)}(-\boldsymbol{k}) = (-1)^{\lambda+1}\hat{\varepsilon}^{(\lambda)}(\boldsymbol{k}). \tag{2.3}$$

Since the magnetic induction is real, the generalised coordinates $q_{\boldsymbol{k}\lambda}$ in equation (2.2) satisfy

$$q^*_{\boldsymbol{k}\lambda} = (-1)^{\lambda+1} q_{-\boldsymbol{k}\lambda}. \tag{2.4}$$

In summing over only two polarisations in equation (2.2) we have ensured that

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.5}$$

which is one of the Maxwell equations saying that there are no magnetic monopoles. With the expansion in equation (2.2) the term involving B^2 in equation (2.1) depends only on the generalised coordinates $q_{\boldsymbol{k}\lambda}$ as would be expected for a potential energy.

A Fourier expansion of the electric field $\boldsymbol{E}$ in the cubic volume V can be made,

$$\boldsymbol{E}(\boldsymbol{x}, t) = \mathrm{i} V^{-1/2} \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} (-1)^{\lambda+1} \hat{\varepsilon}^{(\lambda)}(\boldsymbol{k}) \dot{q}_{\boldsymbol{k}\bar{\lambda}} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}), \tag{2.6}$$

in terms of the generalised velocities, which are written for convenience as $\dot{q}_{\boldsymbol{k}\bar{\lambda}}$. The polarisation index $\bar{\lambda} = -\lambda + 3$, so $\bar{1} = 2$ and $\bar{2} = 1$. The electric field is real because of equation (2.4). Gauss's law

$$\nabla \cdot \boldsymbol{E} = 0 \tag{2.7}$$

is satisfied by the electric field in equation (2.6) because only transverse polarisations are used. Because of the expansion in equation (2.6), the term involving E^2 in equation (2.1) depends only on the generalised velocities, as would be expected for a kinetic energy.

Faraday's law,

$$\nabla \times \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t, \tag{2.8}$$

is satisfied by the electric field in equation (2.6) and the magnetic induction in equation (2.2). The expansion of the magnetic induction includes the wavenumber $k = |\boldsymbol{k}|$, and the expansion of the electric field includes the factor $\mathrm{i}(-1)^{\lambda+1}$ along with the polarisation index $\bar{\lambda}$ for the generalised velocity. These terms are used so that Faraday's law is satisfied identically. In order to derive equation (2.8) we used the relation

$$\hat{k} \times \hat{\varepsilon}^{(\lambda)}(\boldsymbol{k}) = (-1)^{\bar{\lambda}} \hat{\varepsilon}^{(\bar{\lambda})}(\boldsymbol{k}). \tag{2.9}$$

Faraday's law is thus a kinematical equation here, as it is automatically in a gauge theory where potentials are used (Kobe 1978, 1980).

When equations (2.2) and (2.6) are substituted into equation (2.1), the Lagrangian becomes

$$L = \tfrac{1}{2} \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \{\dot{q}^*_{\boldsymbol{k}\lambda} \dot{q}_{\boldsymbol{k}\lambda} - k^2 q^*_{\boldsymbol{k}\lambda} q_{\boldsymbol{k}\lambda}\}, \tag{2.10}$$

which is the Lagrangian for independent harmonic oscillators with complex generalised coordinates. The E^2 term in equation (2.1) is the kinetic energy of the oscillators and the B^2 term is the potential energy of the oscillators. The Euler–Lagrange equations

$$\mathrm{d}(\partial L/\partial \dot{q}^*_{\boldsymbol{k}\lambda})/\mathrm{d}t - \partial L/\partial q^*_{\boldsymbol{k}\lambda} = 0, \tag{2.11}$$

for all $\boldsymbol{k}$ and λ follow from the principle of least action. When equation (2.10) is substituted into equation (2.11), we obtain

$$\ddot{q}_{k\lambda} = -k^2 q_{k\lambda}, \tag{2.12}$$

which is the equation of a harmonic oscillator of frequency $\omega = k$. From equations (2.2), (2.6), and (2.12) we obtain the Ampère–Maxwell law

$$\nabla \times \boldsymbol{B} = \partial \boldsymbol{E}/\partial t \tag{2.13}$$

for the free field. Therefore the Ampère–Maxwell law is a dynamical law, as it is in the case of gauge field theory (Kobe 1978, 1980). Equations (2.5), (2.7), (2.8) and (2.13) are all of the Maxwell equations.

3. Canonical formalism

The Hamiltonian formalism and Poisson brackets are essential for canonical quantisation. In this section we extend the Lagrangian formalism of § 2 to the Hamiltonian formalism. In electromagnetic field theory the Hamiltonian (Sakurai 1967, pp 12–5)

$$H = \tfrac{1}{2} \int \mathrm{d}^3x (E^2 + B^2) \tag{3.1}$$

for the free field is constructed from the Lagrangian in equation (2.1) by the standard canonical method of introducing potentials and treating them as generalised coordinates (Kobe 1981).

We shall use for the free electromagnetic field the Lagrangian L in equation (2.10). From the definition of the conjugate momenta

$$p_{k\lambda} = \partial L/\partial \dot{q}_{k\lambda}, \tag{3.2}$$

the canonical momenta in this case are

$$p_{k\lambda} = \dot{q}^*_{k\lambda}. \tag{3.3}$$

Therefore the Hamiltonian determined by a Legendre transformation on equation (2.10) is (Goldstein 1980)

$$H = \tfrac{1}{2} \sum_{k} \sum_{\lambda=1}^{2} \{p^*_{k\lambda} p_{k\lambda} + k^2 q^*_{k\lambda} q_{k\lambda}\}. \tag{3.4}$$

Equation (3.4) can be shown to be equal to equation (3.1) by using equations (2.2), (2.6) and (3.3).

In terms of the canonical variables $q_{k\lambda}$ and $p_{k\lambda}$, Hamilton's equations are

$$\dot{q}_{k\lambda} = \partial H/\partial p_{k\lambda}, \tag{3.5}$$

and

$$\dot{p}_{k\lambda} = -\partial H/\partial q_{k\lambda}. \tag{3.6}$$

Equations (3.5) and (3.4) give

$$\dot{q}_{k\lambda} = p^*_{k\lambda}, \tag{3.7}$$

which is the complex conjugate of equation (3.3). Equations (3.6) and (3.4) give

$$\dot{p}_{k\lambda} = -k^2 q^*_{k\lambda}, \tag{3.8}$$

which is a dynamical equation. If we eliminate $p_{k\lambda}$ between equations (3.7) and (3.8), we obtain equation (2.12), from which the Ampère–Maxwell law in equation (2.13) is obtained. Equation (3.7) or equation (3.3) allows the electric field in equation (2.6) to be written as

$$\boldsymbol{E}(\boldsymbol{x}, t) = \mathrm{i} V^{-1/2} \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} (-1)^{\lambda+1} \hat{\varepsilon}^{(\lambda)}(\boldsymbol{k}) p^{*}_{\boldsymbol{k}\bar{\lambda}} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}), \tag{3.9}$$

in terms of the conjugate momentum.

The time development of a classical function of the canonical variables can be expressed in terms of the Poisson bracket (Goldstein 1980). Given two functions F and G of $q_{k\lambda}$ and $p_{k\lambda}$, the Poisson bracket is defined by

$$\{F, G\} = \sum_{\boldsymbol{k}} \sum_{\lambda=1}^{2} \left(\frac{\partial F}{\partial q_{k\lambda}} \frac{\partial G}{\partial p_{k\lambda}} - \frac{\partial G}{\partial q_{k\lambda}} \frac{\partial F}{\partial p_{k\lambda}} \right). \tag{3.10}$$

From equation (3.10) we find the Poisson brackets

$$\{q_{k\lambda}, p_{k'\lambda'}\} = \delta_{\boldsymbol{k},\boldsymbol{k}'} \delta_{\lambda,\lambda'}, \tag{3.11}$$

$$\{q_{k\lambda}, q_{k'\lambda'}\} = 0, \tag{3.12}$$

and

$$\{p_{k\lambda}, p_{k'\lambda'}\} = 0, \tag{3.13}$$

where $\delta_{\boldsymbol{k},\boldsymbol{k}'}$ and $\delta_{\lambda,\lambda'}$ are Kronecker deltas.

The time rate of change of a function F of $q_{k\lambda}$, $p_{k\lambda}$ and t is

$$\mathrm{d}F/\mathrm{d}t = \{F, H\} + \partial F/\partial t. \tag{3.14}$$

The equations of motion can be written using equation (3.14) as

$$\dot{q}_{k\lambda} = \{q_{k\lambda}, H\}, \tag{3.15}$$

$$\dot{p}_{k\lambda} = \{p_{k\lambda}, H\}. \tag{3.16}$$

Equation (3.15) gives equation (3.7) when the Hamiltonian in equation (3.4) is used, while equation (3.16) gives equation (3.8). The classical canonical formalism is complete, and the theory can now be quantised in a canonical manner.

The time rate of change of a function F of $q_{k\lambda}$, $p_{k\lambda}$ and t is

4. Canonical quantisation

The process of canonical quantisation involves replacing the generalised coordinates $q_{k\lambda}$ and the conjugate momenta $p_{k\lambda}$ by operators on a Hilbert space (Heitler 1954). The Poisson bracket { , } defined in equation (3.10) is replaced by $(\mathrm{i}\hbar)^{-1}$ times the commutator [,]. Hence equations (3.11)–(3.13) are replaced by

$$[q_{k\lambda}, p_{k'\lambda'}] = \mathrm{i}\hbar \delta_{\boldsymbol{k},\boldsymbol{k}'} \delta_{\lambda,\lambda'}, \tag{4.1}$$

$$[q_{k\lambda}, q_{k'\lambda'}] = 0, \tag{4.2}$$

and

$$[p_{k\lambda}, p_{k'\lambda'}] = 0, \tag{4.3}$$

respectively.

If F is a function of the generalised coordinates $q_{k\lambda}$ and the conjugate momenta $p_{k\lambda}$, it becomes an operator on quantisation. The order of the $q_{k\lambda}$ and $p_{k\lambda}$ makes a difference and a symmetric form which is Hermitian is usually used. The equation of motion for F in equation (3.14) becomes on quantisation

$$\mathrm{d}F/\mathrm{d}t = (\mathrm{i}\hbar)^{-1}[F, H] + \partial F/\partial t, \tag{4.4}$$

where the partial derivative indicates a differentiation with respect to the explicit dependence of F on time. Equations (2.2), (2.4), (3.9) and (3.4) hold as operator equations with complex conjugation (*) replaced by Hermitian conjugation (†).

We can use equations (4.1)–(4.3) to find commutation relations among the components of $\boldsymbol{B}$ and $\boldsymbol{E}$. In performing the calculations, a useful relation is

$$\sum_{\lambda=1}^{2} \varepsilon_i^{(\lambda)}(\boldsymbol{k})\varepsilon_j^{(\lambda)}(\boldsymbol{k}) + k_i k_j/k^2 = \delta_{ij}, \tag{4.5}$$

where the Cartesian components are denoted by the subscripts i and j. Equation (4.5) results from the fact that $\hat{\varepsilon}^{(1)}(\boldsymbol{k})$, $\hat{\varepsilon}^{(2)}(\boldsymbol{k})$, $\hat{k}$ form an orthonormal basis. Equations (4.1)–(4.3) and (4.5) give the gauge-invariant equal-time commutation relations for the components of $\boldsymbol{B}$ in equation (2.2) and $\boldsymbol{E}$ in equation (3.9),

$$[B_i(\boldsymbol{x}, t), E_j(\boldsymbol{x}', t)] = \mathrm{i}\hbar\varepsilon_{ijk}\, \partial_k \delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{4.6}$$

$$[B_i(\boldsymbol{x}, t), B_j(\boldsymbol{x}', t)] = 0, \tag{4.7}$$

and

$$[E_i(\boldsymbol{x}, t), E_j(\boldsymbol{x}', t)] = 0, \tag{4.8}$$

where ε_{ijk} is the Levi–Civita symbol, the derivative is $\partial_k = \partial/\partial x^k$, and summation over repeated indices from 1 to 3 is implied. The commutation relations in equations (4.6)–(4.8) are the same as obtained by other means of quantisation (Heitler 1954).

5. A canonical transformation on Fourier coefficients

The canonical variables used to describe a physical system are not unique. As long as Hamilton's equations in equations (3.5) and (3.6) are unchanged under a change of canonical variables to $q'_{k\lambda}$ and $p'_{k\lambda}$, the new set of canonical variables describes the same physical system as $q_{k\lambda}$ and $p_{k\lambda}$ (Goldstein 1980).

Consider the transformation, parametrised by θ, to new variables $q'_{k\lambda}$ and $p'_{k\lambda}$,

$$q_{k\lambda} = q'_{k\lambda} \cos\theta + \alpha(\bar{\lambda}) p'^{*}_{k\bar{\lambda}} \sin\theta, \tag{5.1}$$

and

$$p_{k\lambda} = \beta(\bar{\lambda}) q'^{*}_{k\bar{\lambda}} \sin\theta + p'_{k\lambda}\cos\theta, \tag{5.2}$$

where $\alpha(\bar{\lambda})$ and $\beta(\bar{\lambda})$ are introduced for dimensional reasons and $\bar{\lambda} = -\lambda + 3$. Under certain conditions on α and β, equations (5.1) and (5.2) describe a rotation in a generalised complex phase space. The Hamiltonian in terms of $q'_{k\lambda}$ and $p'_{k\lambda}$ is

$$H'(q', p') = H(q(q', p'), p(q', p')). \tag{5.3}$$

By definition, the transformation of equations (5.1) and (5.2) is canonical if equations (3.5) and (3.6) imply that

$$\dot{q}'_{k\lambda} = \partial H'/\partial p'_{k\lambda} \tag{5.4}$$

and

$$\dot{p}'_{k\lambda} = -\partial H'/\partial q'_{k\lambda}. \tag{5.5}$$

If the chain rule and equations (3.5) and (3.6) are used in differentiating equation (5.3), we obtain

$$\partial H'/\partial q'_{k\lambda} = -\dot{p}'_{k\lambda}(\cos^2\theta - \alpha(\lambda)\beta(\lambda)\sin^2\theta) - (\beta(\bar{\lambda})+\beta(\lambda))\dot{q}'^{*}_{k\bar{\lambda}}\sin\theta\cos\theta, \tag{5.6}$$

and

$$\partial H'/\partial p'_{k\lambda} = \dot{q}'_{k\lambda}(\cos^2\theta - \beta(\lambda)\alpha(\lambda)\sin^2\theta) + (\alpha(\lambda)+\alpha(\bar{\lambda}))\dot{p}'^{*}_{k\bar{\lambda}}\sin\theta\cos\theta. \tag{5.7}$$

Equations (5.6) and (5.7) imply that the necessary and sufficient conditions for equations (5.4) and (5.5) to hold are

$$\alpha(\lambda)\beta(\lambda) = -1, \tag{5.8}$$

$$\alpha(\lambda)+\alpha(\bar{\lambda}) = 0, \tag{5.9}$$

and

$$\beta(\lambda)+\beta(\bar{\lambda}) = 0. \tag{5.10}$$

Since $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same dimensions, we see from a comparison of equations (2.2) and (3.9) that $\beta(\lambda)$ has the dimensions of k and $\alpha(\lambda)$ has the dimensions of k^{-1}. Therefore, under the transformation in equations (5.1) and (5.2) Hamilton's equations are unchanged in form if equations (5.8)–(5.10) are also satisfied.

6. Duality transformations

In this section, we show that a duality transformation on the electric and magnetic fields is produced by the canonical transformation of § 5 on the Fourier coefficients. In equations (5.1) and (5.2), we choose

$$\alpha(\lambda) = (-1)^{\lambda+1}\mathrm{i}k^{-1} \tag{6.1}$$

and

$$\beta(\lambda) = (-1)^{\lambda+1}\mathrm{i}k. \tag{6.2}$$

It is then seen from equations (6.1) and (6.2) that equations (5.8)–(5.10) are satisfied. Hence equations (5.6) and (5.7) reduce to equations (5.4) and (5.5). Equations (5.1) and (5.2) constitute a canonical transformation when equations (6.1) and (6.2) are used. This canonical transformation produces a duality transformation on $\boldsymbol{E}$ and $\boldsymbol{B}$.

A new magnetic induction vector $\boldsymbol{B}'$ can be defined in analogy with equation (2.2) as

$$\boldsymbol{B}'(\boldsymbol{x}, t) = V^{-1/2}\sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2} k\hat{\varepsilon}^{(\lambda)}(\boldsymbol{k})q'_{k\lambda}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}), \tag{6.3}$$

where $q'_{k\lambda}$ are generalised coordinates. A new electric field vector $\boldsymbol{E}'$ can also be defined in analogy with equation (3.9) as

$$\boldsymbol{E}'(\boldsymbol{x}, t) = \mathrm{i}V^{-1/2}\sum_{\boldsymbol{k}}\sum_{\lambda=1}^{2}(-1)^{\lambda+1}\varepsilon^{(\lambda)}(\boldsymbol{k})p'^{*}_{k\bar{\lambda}}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}), \tag{6.4}$$

where $p'_{k\lambda}$ are new canonical momenta. If equation (5.1) with equation (6.1) is substituted into equation (2.2), the result is

$$\boldsymbol{B} = \boldsymbol{B}' \cos\theta - \boldsymbol{E}' \sin\theta, \tag{6.5}$$

with the help of equation (3.9). Likewise, if equation (5.2) with equation (6.2) is substituted into equation (3.9), the result is

$$\boldsymbol{E} = \boldsymbol{B}' \sin\theta + \boldsymbol{E}' \cos\theta. \tag{6.6}$$

Equations (6.5) and (6.6) are the usual duality transformations on the classical electromagnetic field (Jackson 1975, p 252). The duality transformation remains valid for the quantised electromagnetic field where the generalised coordinates and momenta in equations (6.3) and (6.4) are replaced by the corresponding operators that satisfy the canonical commutation relations.

When equations (6.5) and (6.6) for the quantised fields are used in equations (4.6)–(4.8), we find

$$[B'_i(\boldsymbol{x}, t), E'_j(\boldsymbol{x}', t)] = \mathrm{i}\hbar\varepsilon_{ijk}\partial_k\delta(\boldsymbol{x}-\boldsymbol{x}'), \tag{6.7}$$

$$[B'_i(\boldsymbol{x}, t), B'_j(\boldsymbol{x}', t)] = 0, \tag{6.8}$$

and

$$[E'_i(\boldsymbol{x}, t), E'_j(\boldsymbol{x}', t)] = 0. \tag{6.9}$$

Therefore the form of the commutation relations of the theory is unchanged under duality transformation. The Maxwell equations for the free fields in equations (2.5), (2.7), (2.8) and (2.13) are also form invariant under the duality transformation in equations (6.5) and (6.6).

For the choice of $\theta = \pi/2$ in equations (6.5) and (6.6), we obtain

$$\boldsymbol{B} = -\boldsymbol{E}' \tag{6.10}$$

and

$$\boldsymbol{E} = \boldsymbol{B}', \tag{6.11}$$

which essentially reverses the roles of the electric and magnetic fields. The Maxwell equations and the commutation relations are form invariant under general duality transformations, and are therefore unchanged under the transformation in equations (6.10) and (6.11). From equations (5.1) and (5.2) the roles of generalised coordinates and conjugate momenta are interchanged when $\theta = \pi/2$. In this case the Fourier coefficients of $\boldsymbol{E}'$ are generalised coordinates and the coefficients of $\boldsymbol{B}'$ are canonical momenta, which is the choice made by Scully and Lamb (1967) and Sargent *et al* (1974). Therefore the theory of Scully and Lamb for the *free* electromagnetic field is equivalent to the theory of Power (1964), which is developed here in a canonical way, since they are related by a duality transformation.

7. Conclusion

In this paper a manifestly gauge-invariant canonical quantisation of the free electromagnetic field is made using the standard Lagrangian density for the electromagnetic field $\frac{1}{2}(E^2 - B^2)$. If the Lagrangian of the free electromagnetic field is compared with the Lagrangian of classical mechanics, it is natural to associate $\frac{1}{2}B^2$ with the potential energy and $\frac{1}{2}E^2$ with the kinetic energy. The Fourier coefficients of $\boldsymbol{B}$ are associated with generalised coordinates, and the Fourier coefficients of $\boldsymbol{E}$ are associated with generalised velocities, which is consistent with the Lagrangian as kinetic energy minus potential energy. In this procedure Faraday's law is kinematical, while the Ampère–Maxwell law is a dynamical law, which is consistent with the gauge theory point of view (Kobe 1978, 1980). This choice, which is used by Power (1964), is also consistent with the usual radiation gauge quantisation procedure (Heitler 1954, Sakurai 1967).

On the other hand, Scully and Lamb (1967) consider only the Hamiltonian density $\frac{1}{2}(E^2+B^2)$, where there is no suggestion as to how the kinetic and potential energies are distributed. They identify the Fourier coefficients of $\boldsymbol{E}$ with generalised coordinates and the Fourier coefficients of $\boldsymbol{B}$ with generalised momenta. In doing so, they chose $\frac{1}{2}B^2$ to be the kinetic energy and $\frac{1}{2}E^2$ to be the potential energy. They could then quantise the system in a canonical way. In their theory Faraday's law is the dynamical law, while the Ampère–Maxwell law *without sources* is the kinematical law. This approach is not consistent with the gauge theory point of view (Kobe 1978, 1980). It fails if interactions are included, because currents are not included in the Ampère–Maxwell law.

This paper shows that when the Lagrangian is used, as it must be to define canonical momenta, the theory of Power is based on the standard Lagrangian density $\frac{1}{2}(E^2-B^2)$ while the theory of Scully and Lamb is based on the non-standard Lagrangian density $\frac{1}{2}(B^2-E^2)$. For the *free* electromagnetic field the procedure of Scully and Lamb can be obtained from the procedure of Power by a canonical transformation on the generalised coordinates and momenta, which is a rotation in generalised phase space. We show that this canonical transformation is equivalent to a duality transformation on the electric and magnetic fields.

When the electromagnetic field interacts with charged matter this equivalence of the procedure of Power and Scully and Lamb no longer holds. In this case the procedure of Power based on the standard Lagrangian density and developed in a canonical way in this paper must be used. The interaction of the quantised electromagnetic field with charged matter can be taken into account by using the multipolar Hamiltonian, which expresses the Hamiltonian in terms of integrals involving the electric and magnetic fields (Woolley 1971, 1974, 1975, Babiker *et al* 1974). On the other hand, the minimally coupled Hamiltonian with the transverse vector potential can also be used, since the transverse vector potential can be written as an integral involving the magnetic induction (Belinfante 1951a, b, Belinfante and Lomont 1951, Marx 1970, 1972). The multipolar Hamiltonian is related to the minimally coupled Hamiltonian using the transverse vector potential by a unitary transformation (Woolley 1971, 1974, 1975, Power 1978). There are thus ways in which the interaction between the particles and the electromagnetic field can be expressed in terms of the electric and magnetic fields, but they are non-local. Therefore potentials, and the consequent problems of fixing the gauge and proving gauge invariance, can be avoided, since the electromagnetic field can be quantised in a completely canonical gauge-invariant way using only the magnetic and electric fields.

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